Symbolic Differentiation Optimization

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Outline

- Symbolic Differentiation and its improvement
- Optimization of Derivative Evaluation

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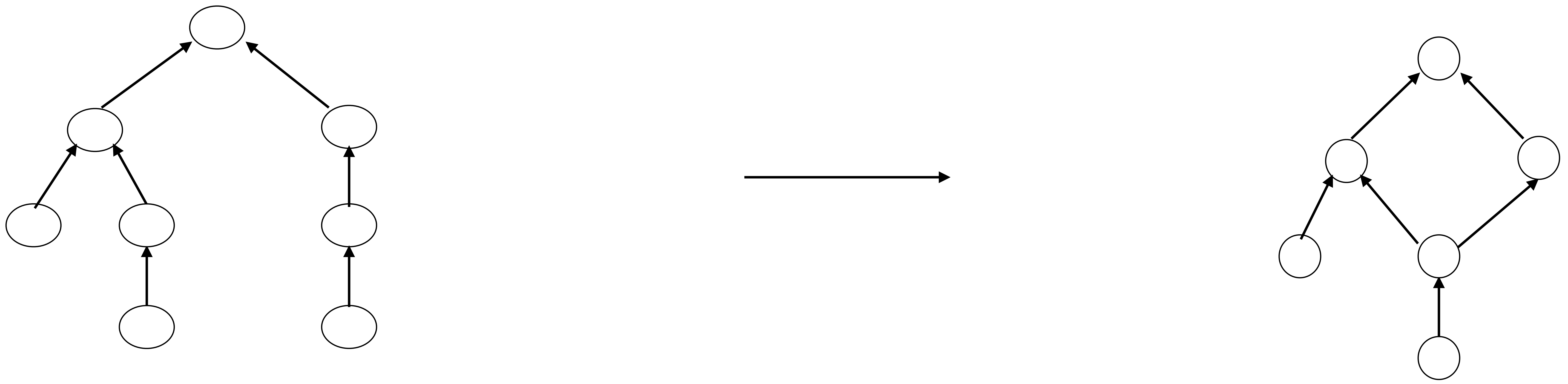
- **Symbolic Differentiation and its improvement**
- Optimization of Derivative Evaluation

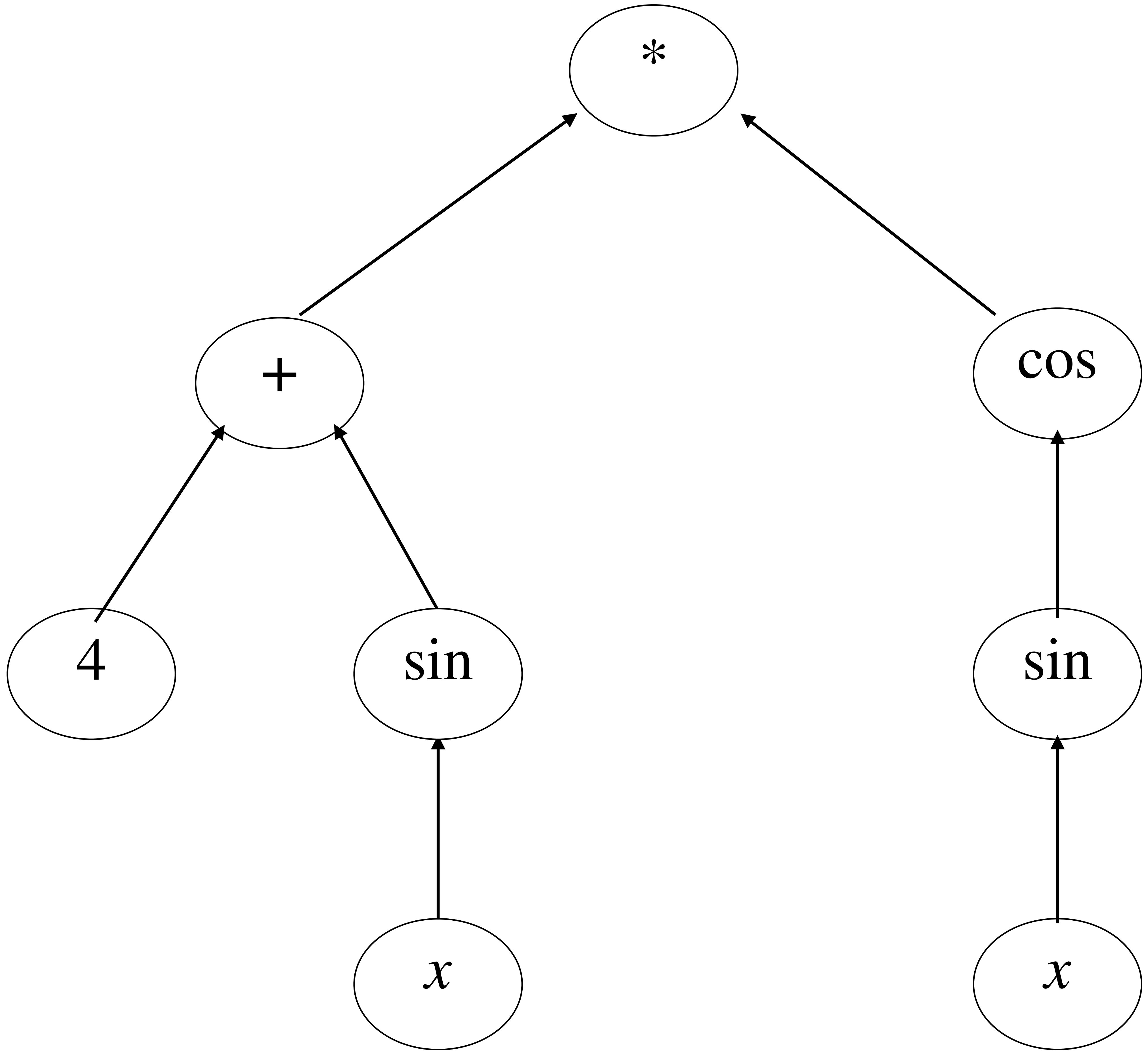
Improvement of Symbolic Differentiation

- We can find analogies between Symbolic Differentiation and Automatic Differentiation through the improvement of Symbolic one. [Soeren, 2020]



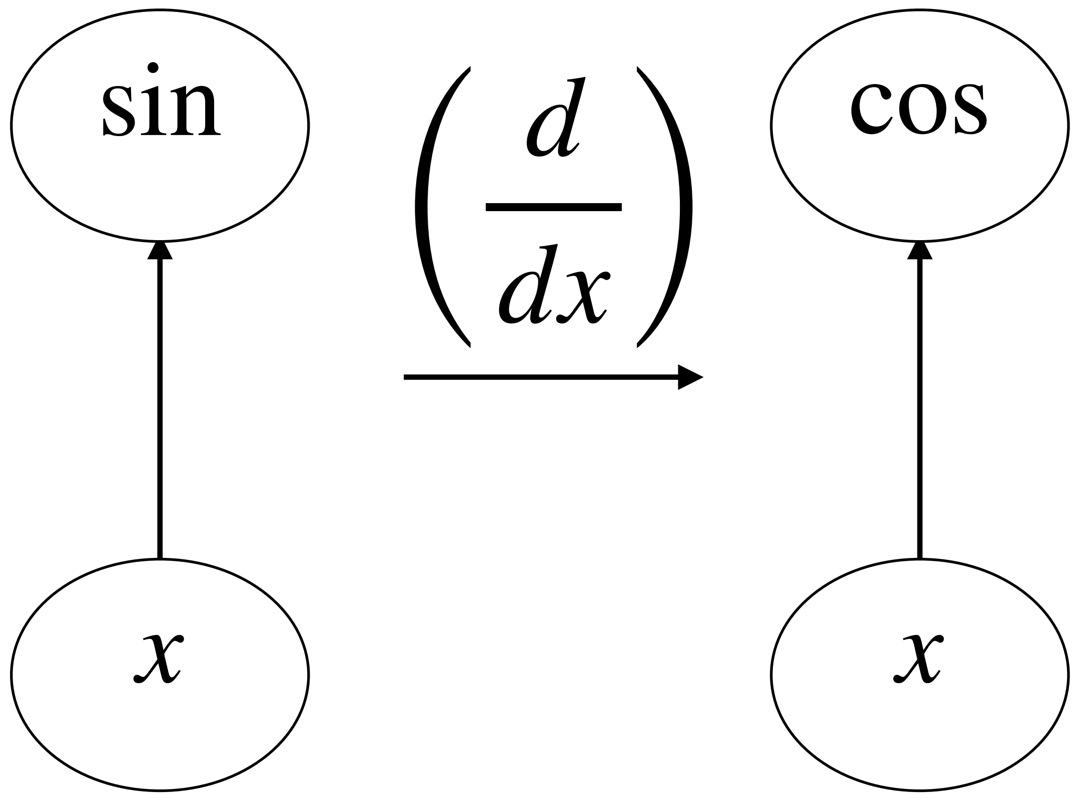
Expression Tree

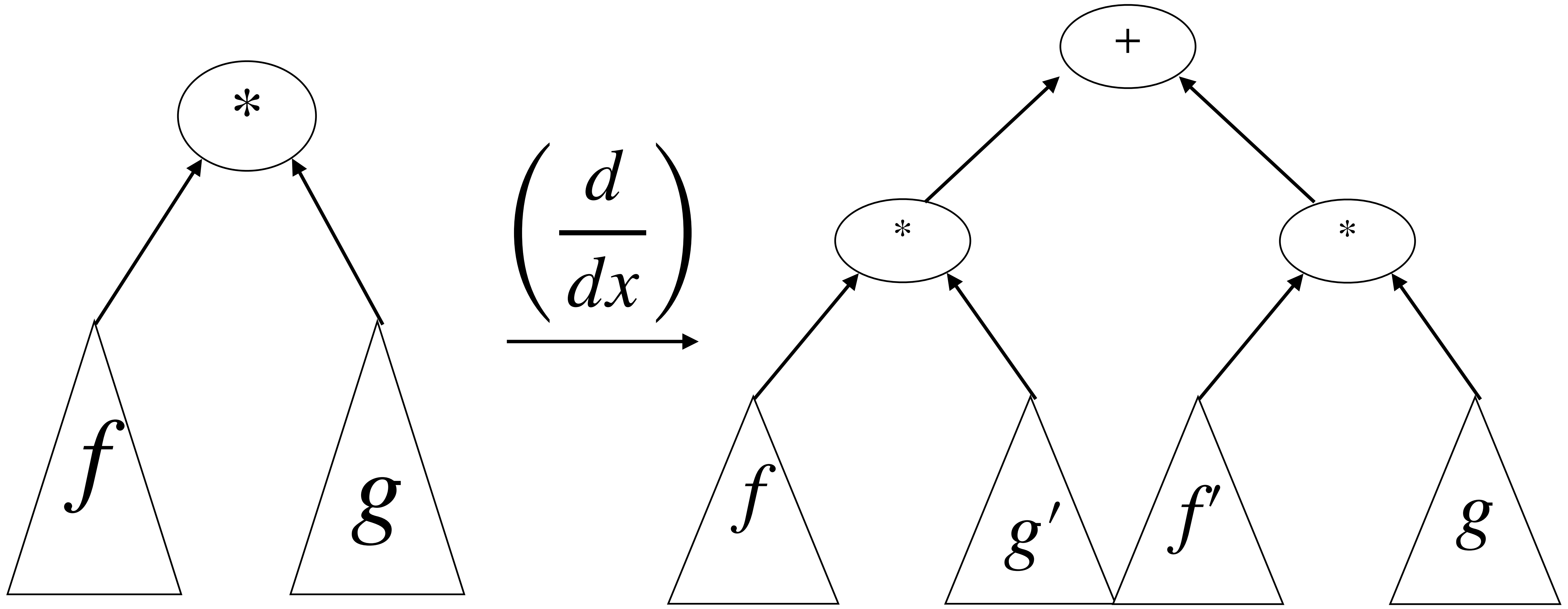
$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x)) =$

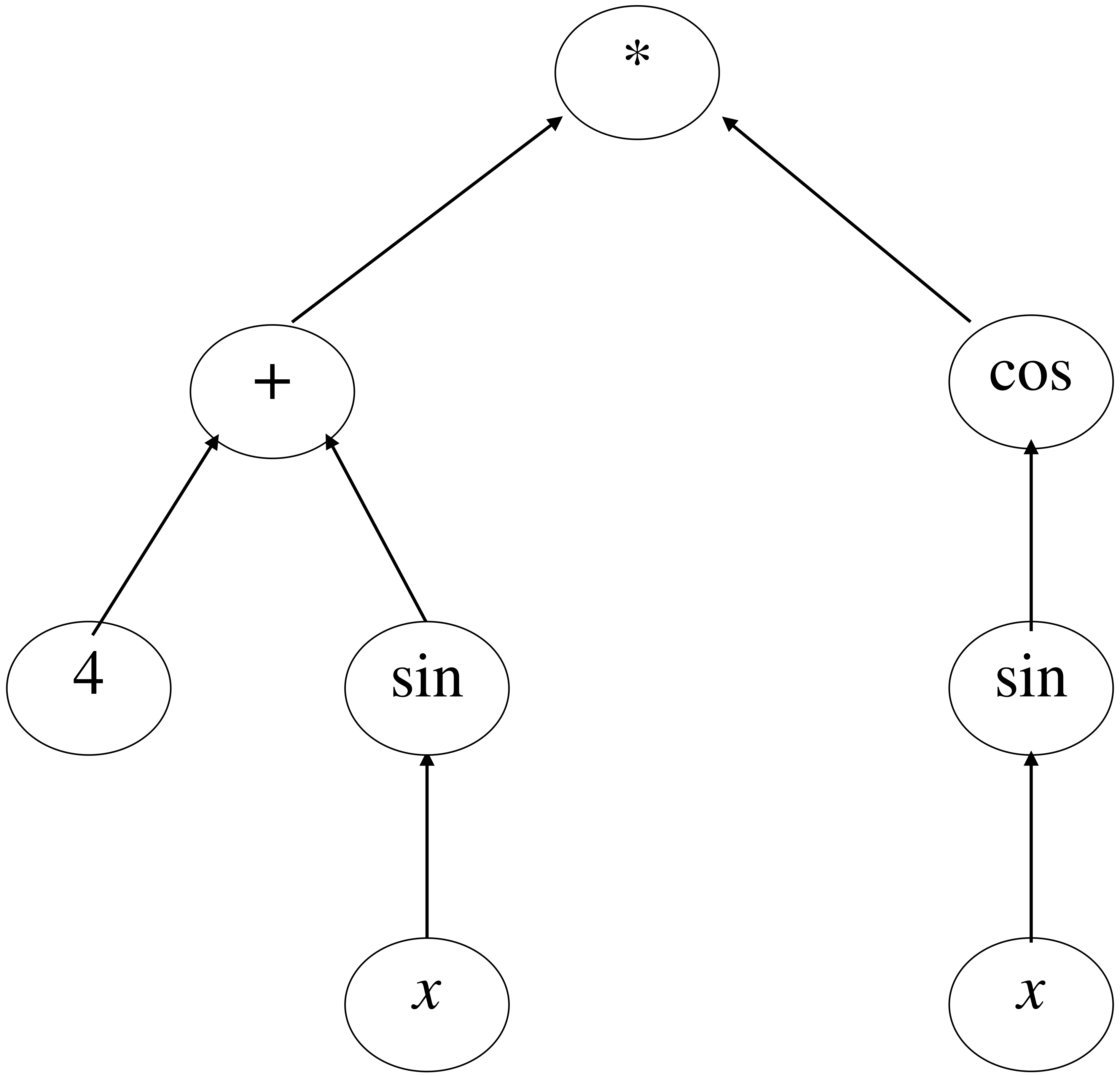


Symbolic Differentiation

$$\left(\frac{d}{dx}\right)\big((4+\sin(x))\cdot\cos(\sin(x))\big)$$



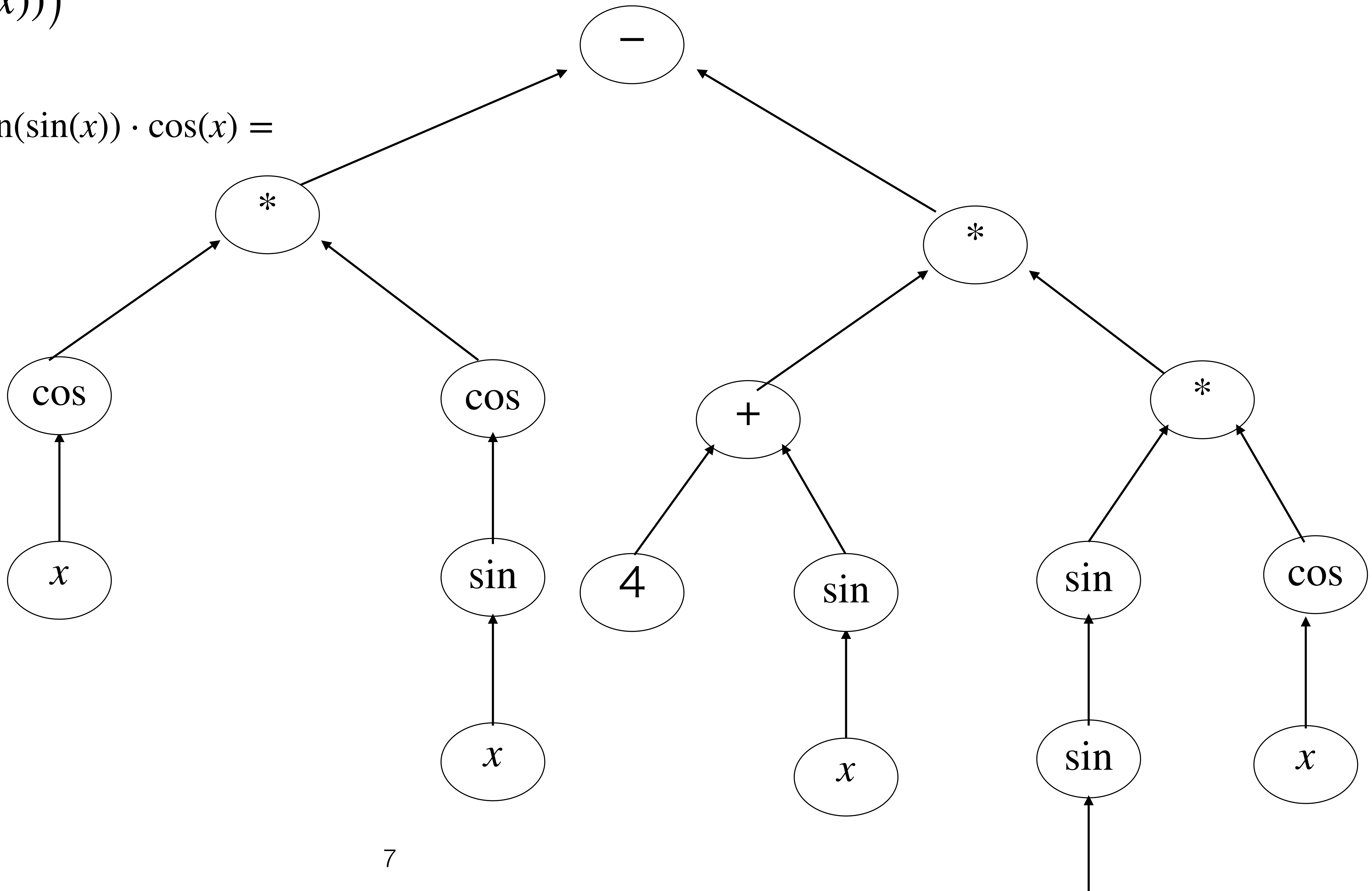




Symbolic Differentiation

$$\left(\frac{d}{dx}\right)\big((4+\sin(x))\cdot\cos(\sin(x))\big)$$

$$= \cos(x)\cdot\cos(\sin(x)) - (4+\sin(x))\cdot\sin(\sin(x))\cdot\cos(x) =$$



Symbolic Differentiation

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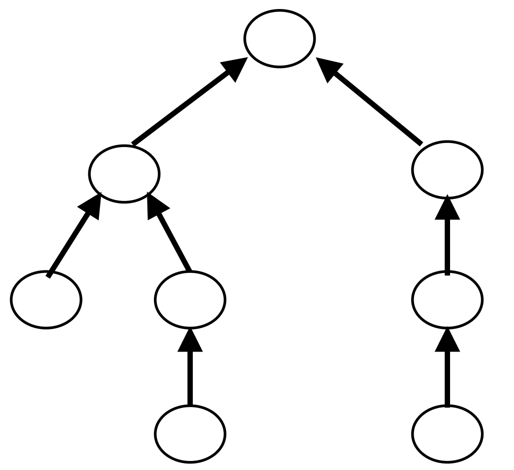
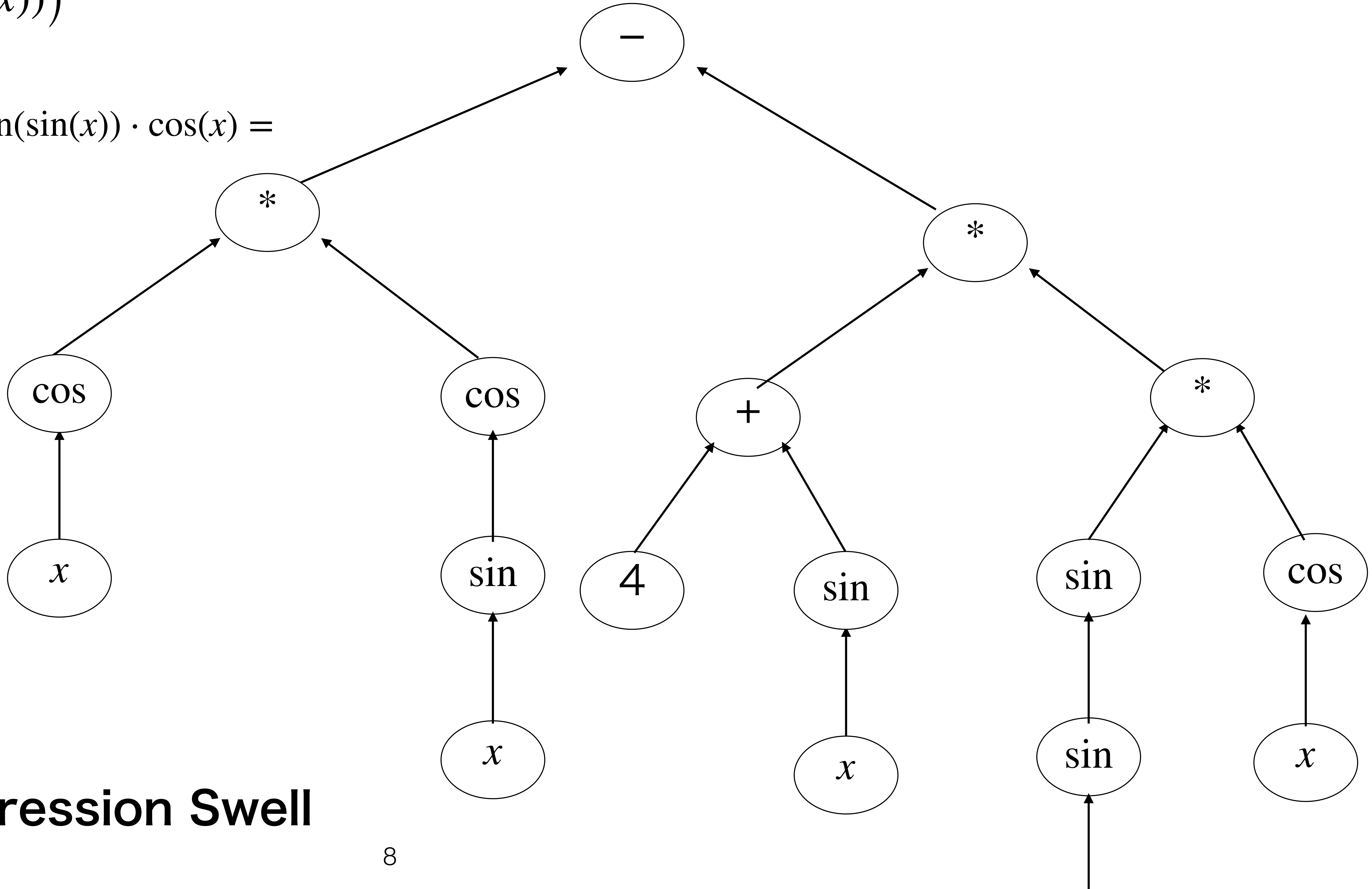


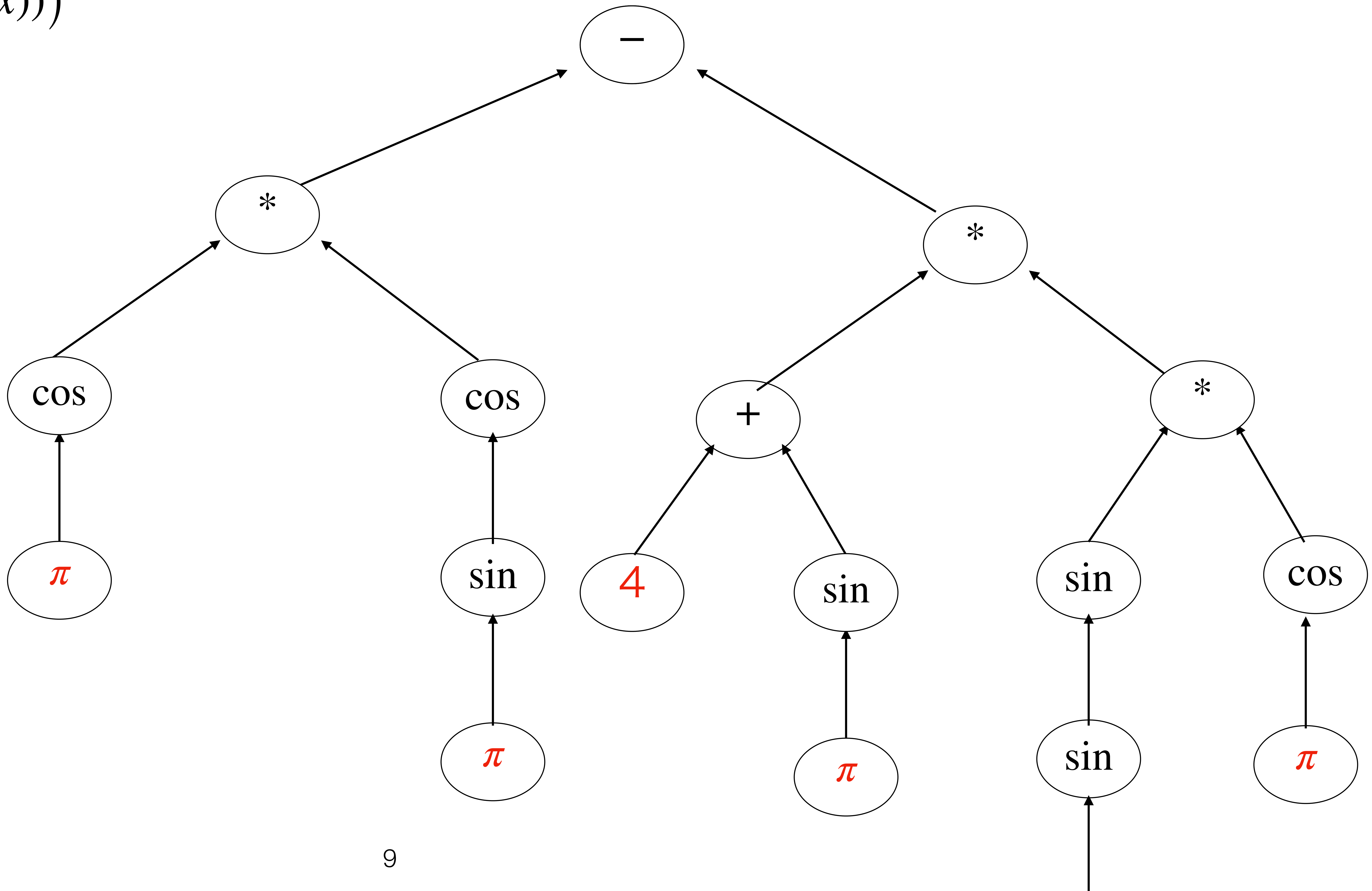
original expression



called as Expression Swell

Evaluation of Derivative on Expression Tree$(x = \pi)$

$$\left(\frac{d}{dx}\right)\left((4+\sin(x))\cdot\cos(\sin(x))\right)$$



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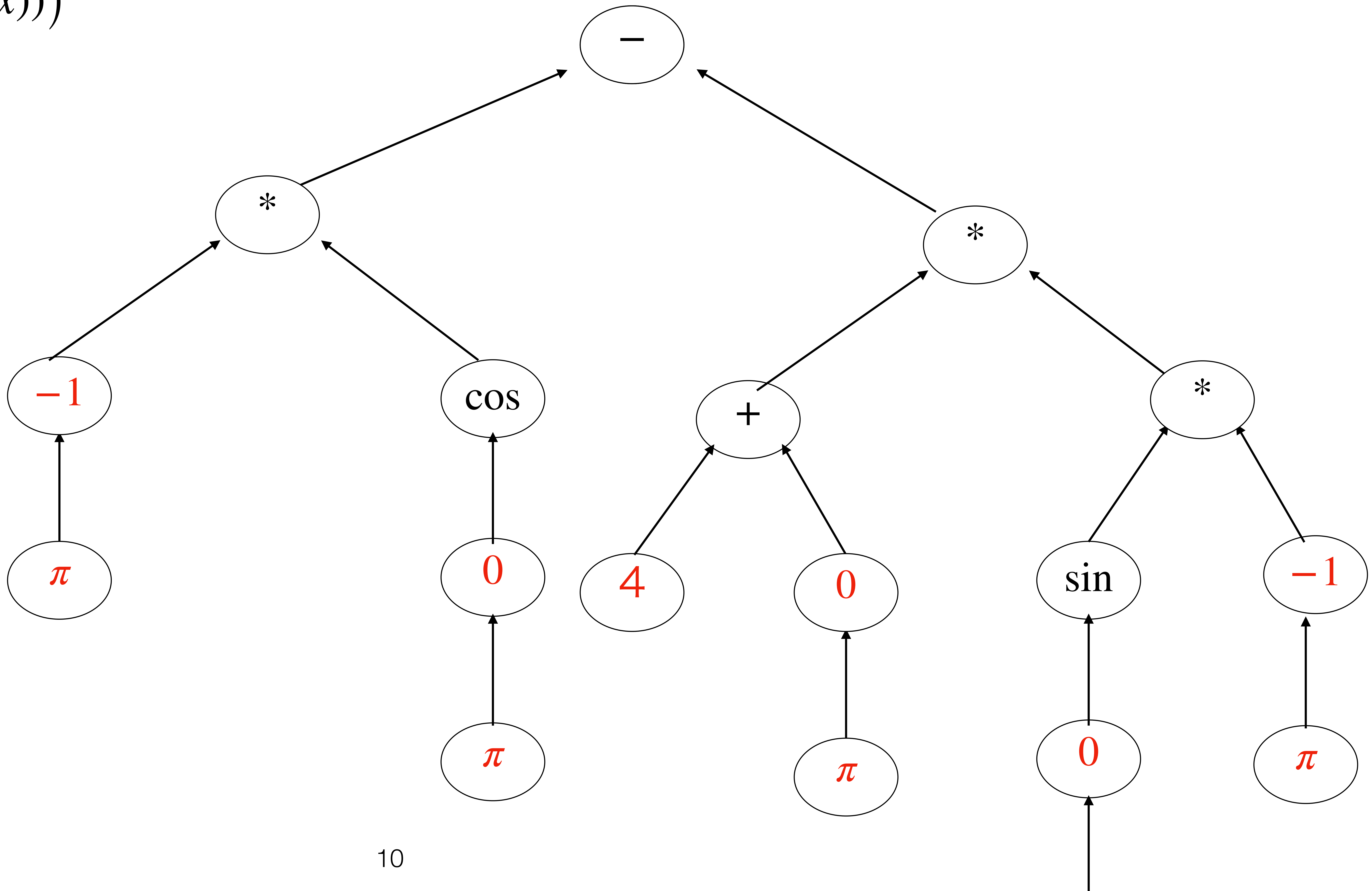

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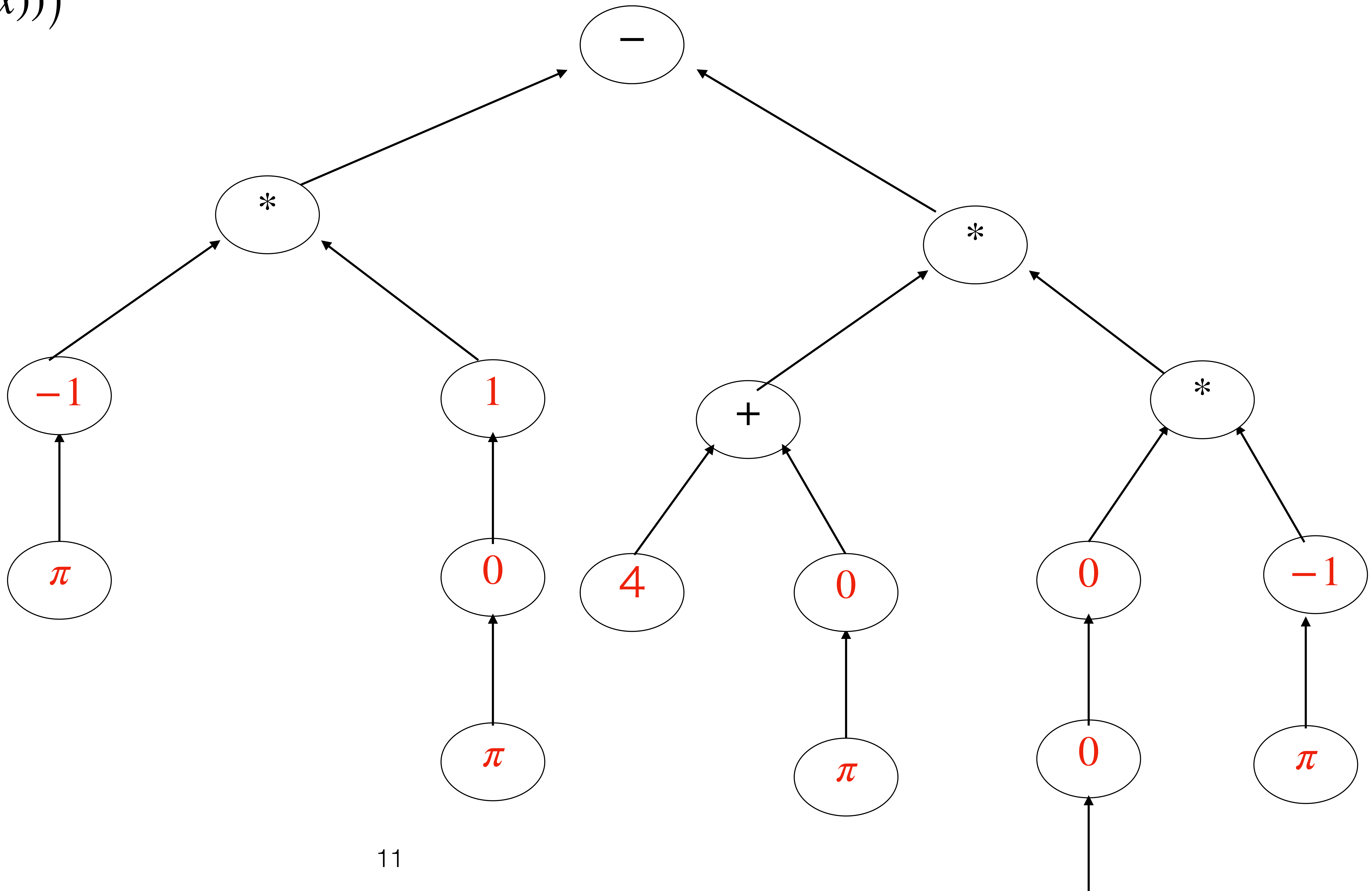

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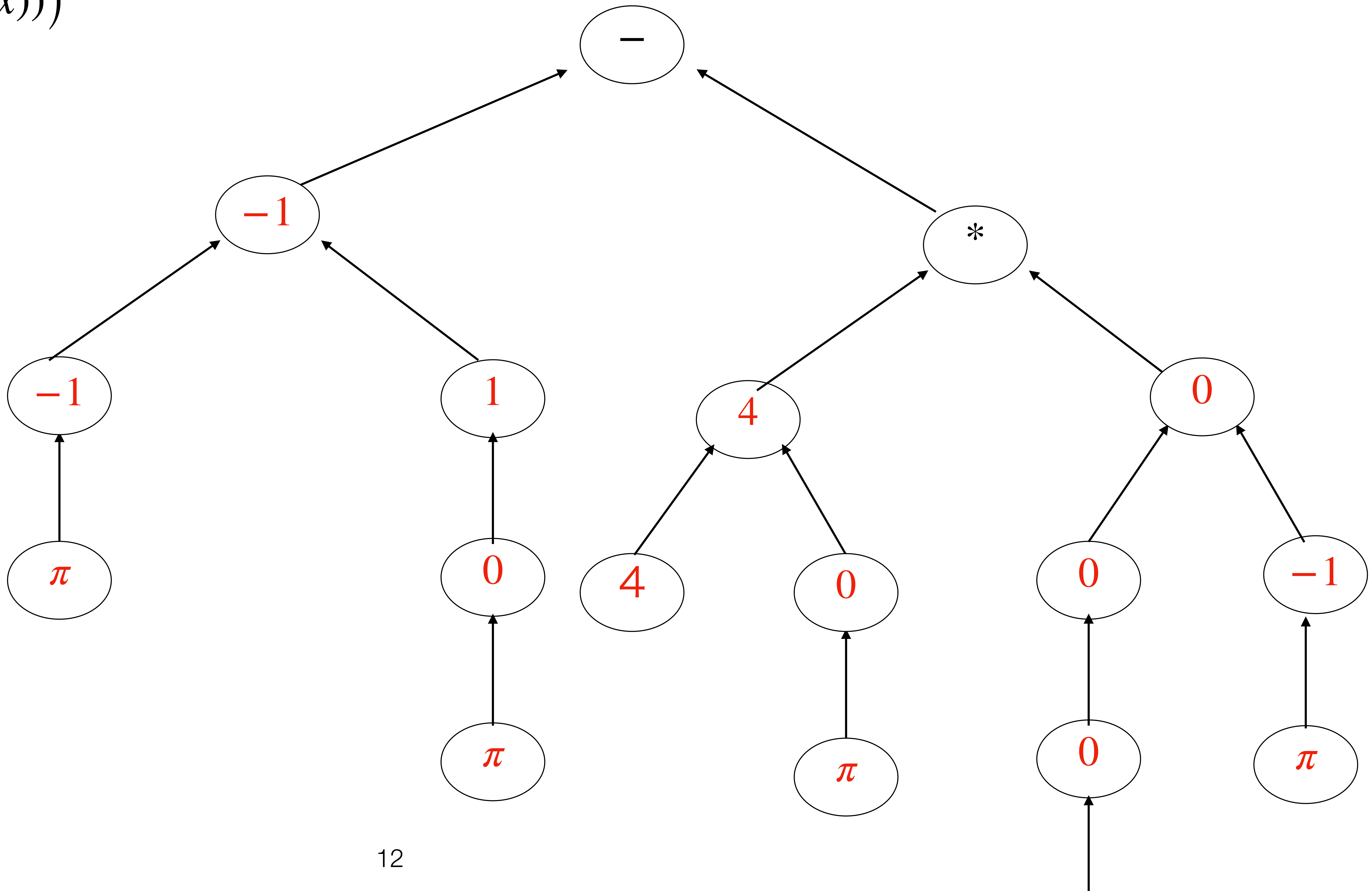
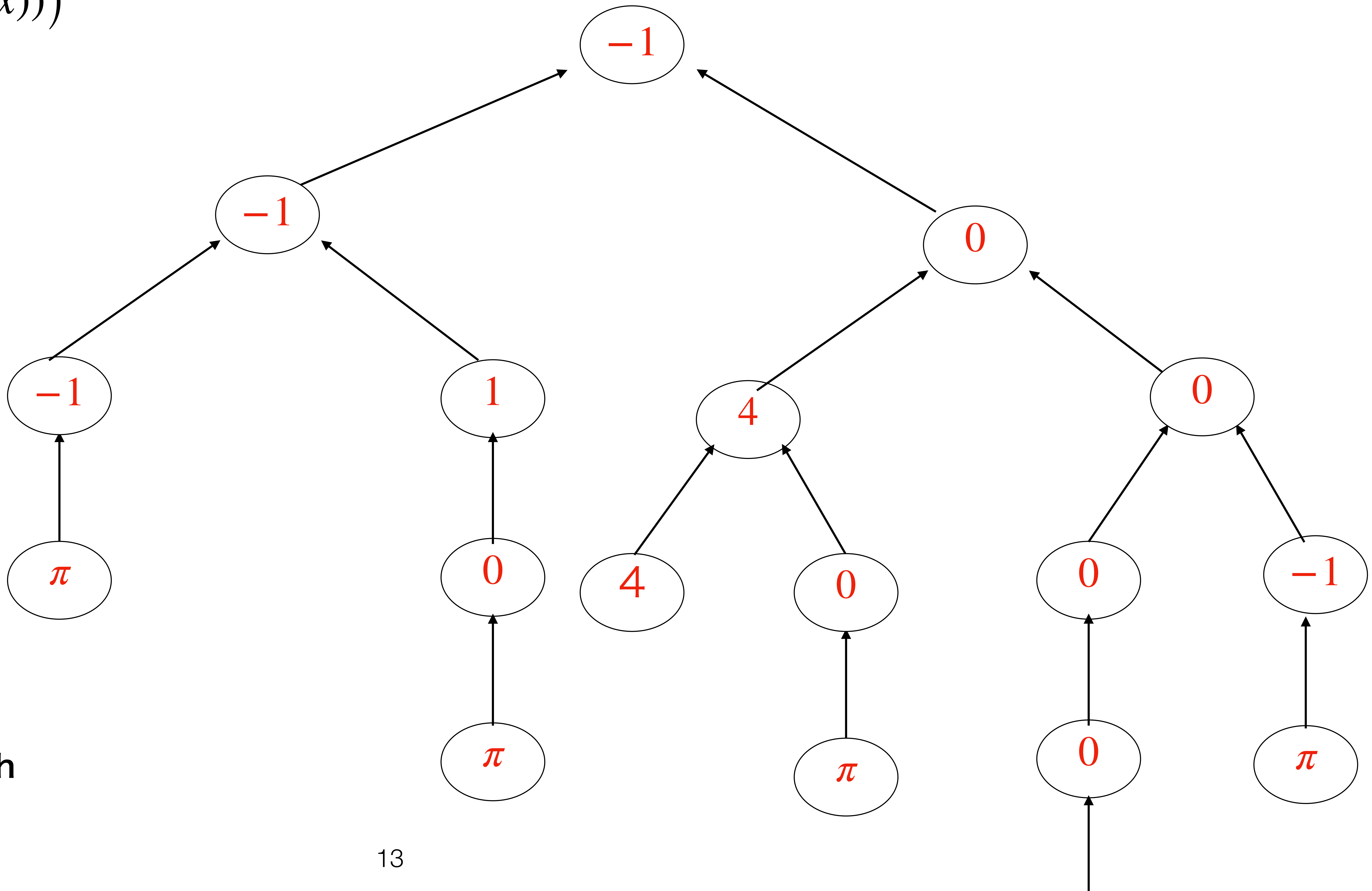


Evaluation of Derivative on Expression Tree($x = \pi$)

$$\left(\frac{d}{dx}\right)\left((4+\sin(x))\cdot\cos(\sin(x))\right)$$

$$= -1$$

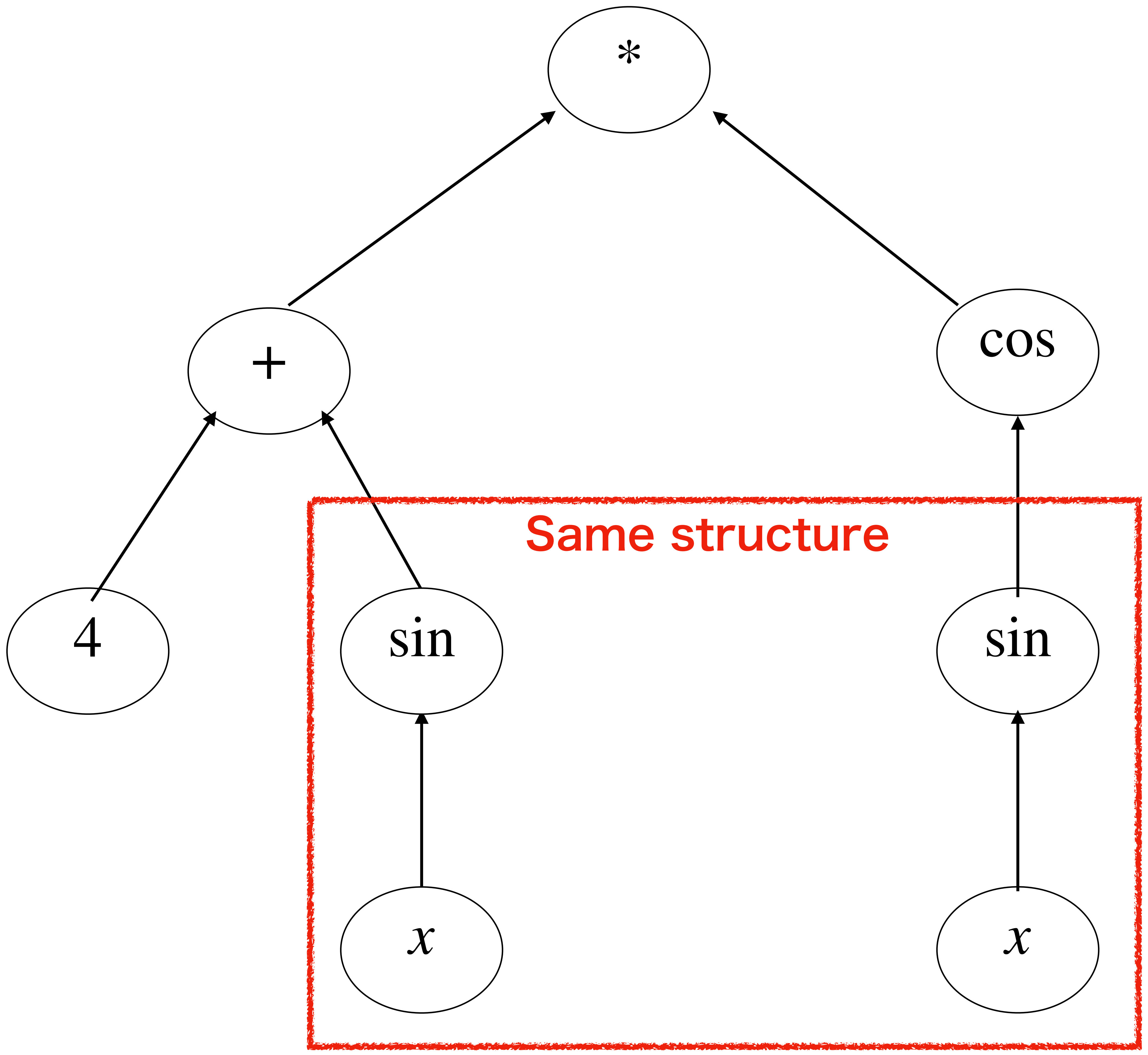


→Let's improve !

1. Tree to DAG
2. Introduction of Derivative Graph

1. Tree to DAG(Directed Acyclic Graph)

$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x))$



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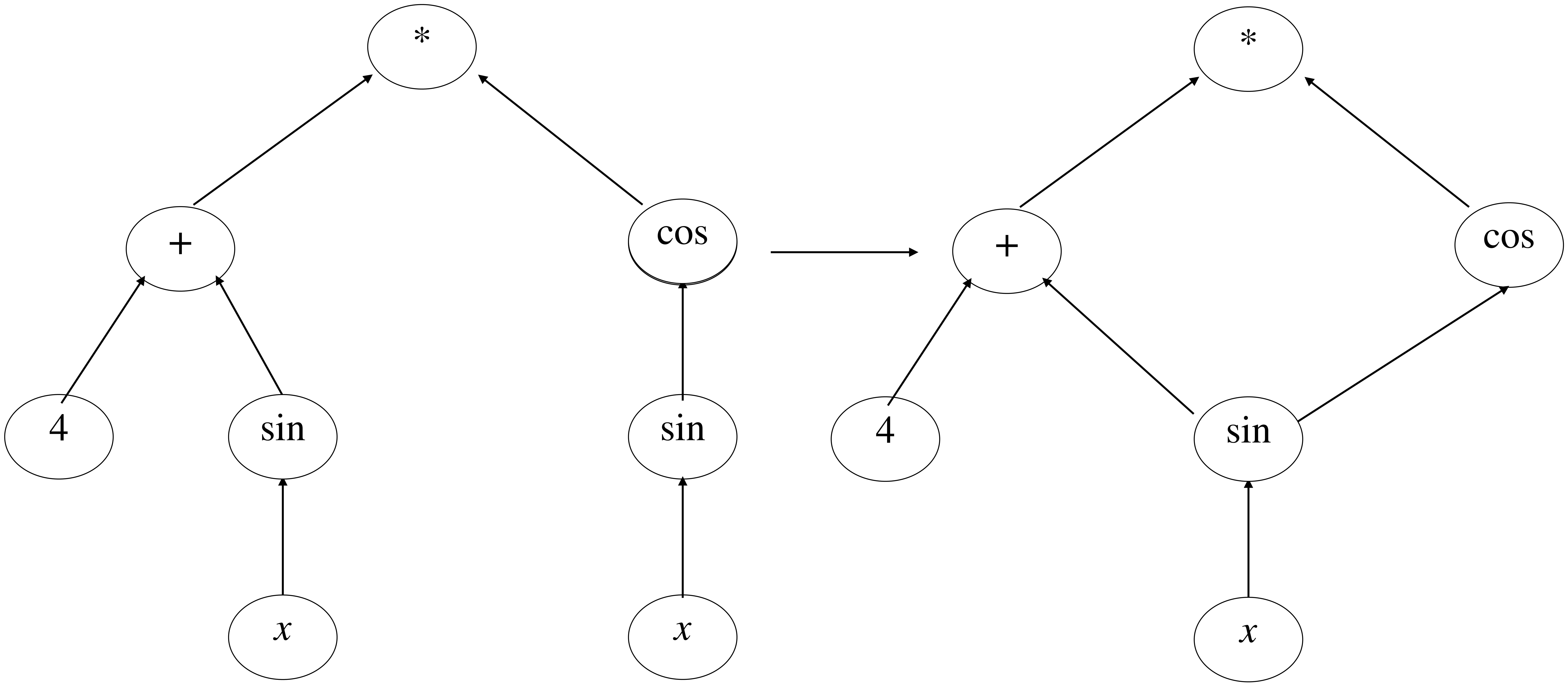
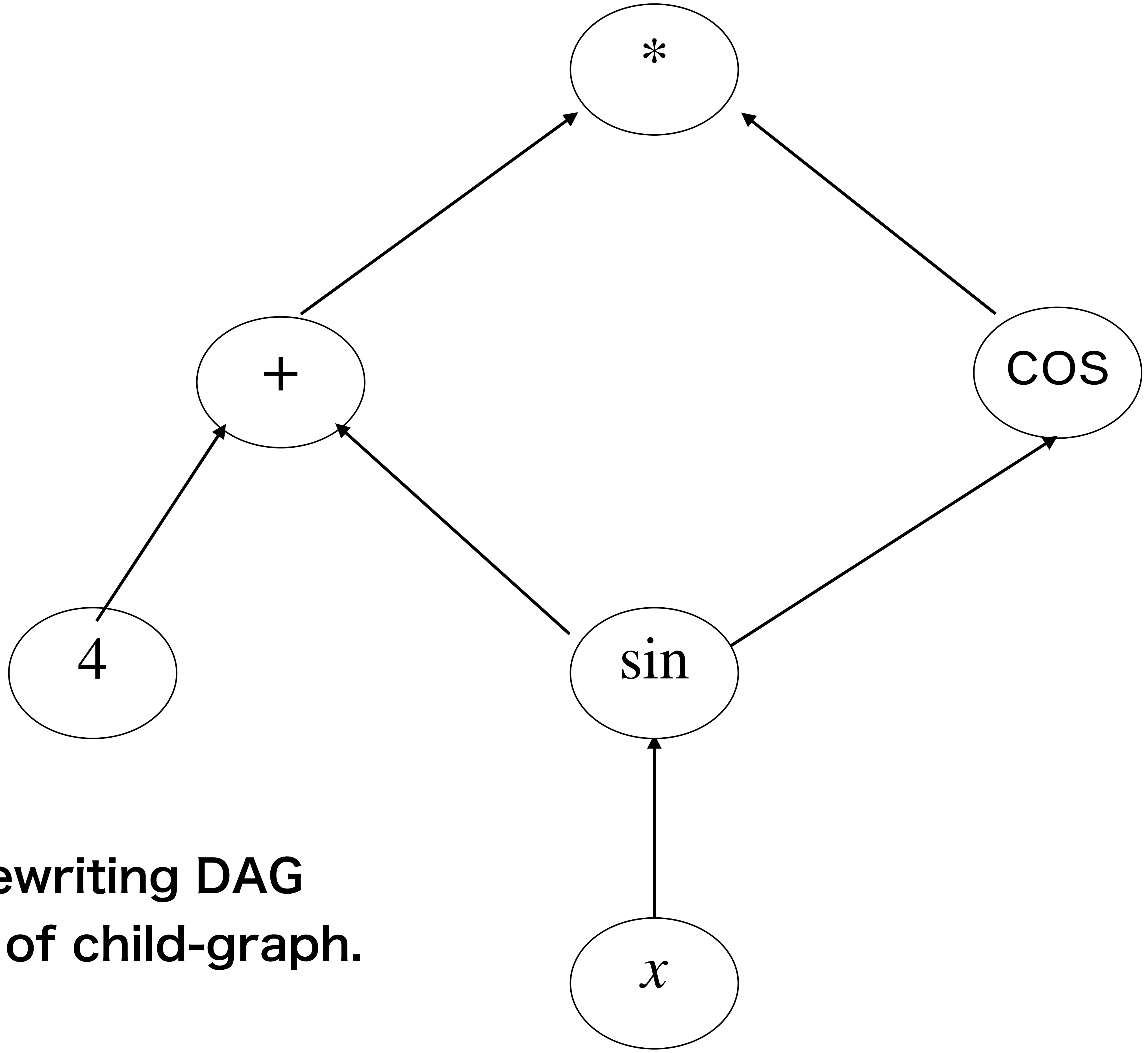


2. Introduction of Derivative graph

$$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x)) = a(x) \cdot b(x)$$



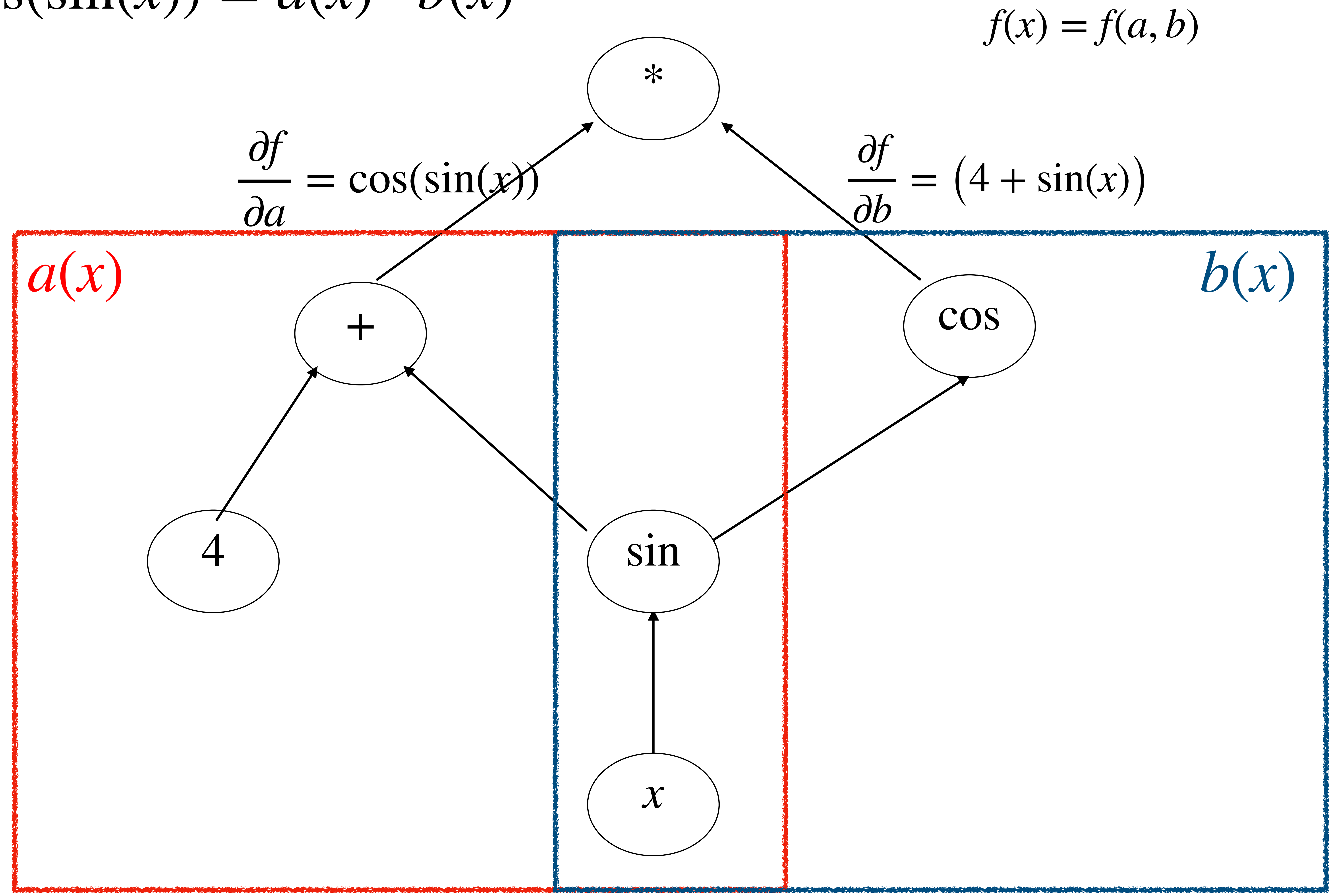
Use new data structure instead of rewriting DAG
Let edges have derivative expressions of child-graph.

2. Introduction of Derivative graph

$$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x)) = a(x) \cdot b(x)$$

$$a(x) = 4 + \sin(x)$$

$$b(x) = \cos(\sin(x))$$



2. Introduction of Derivative graph

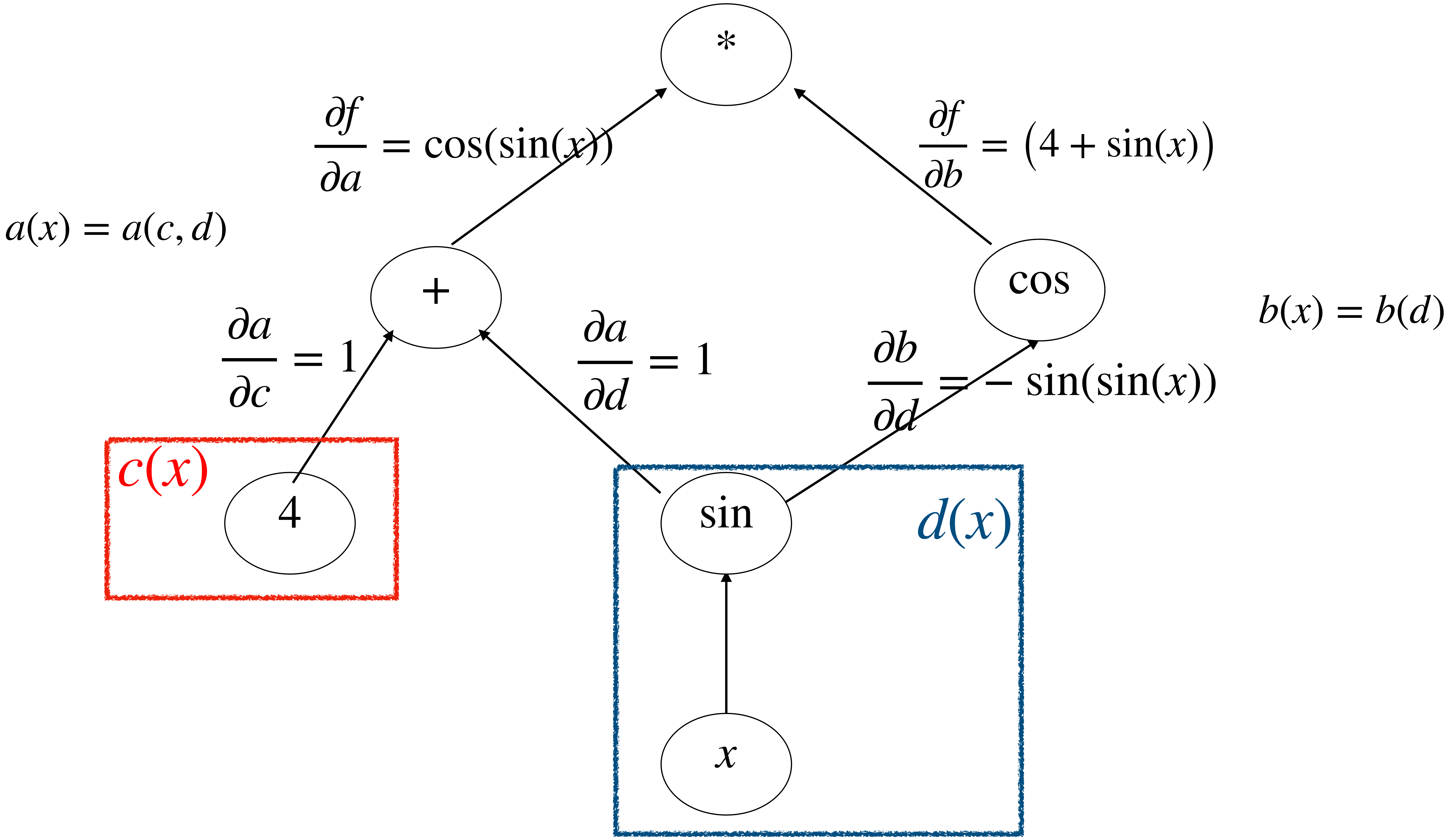
$$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x)) = a(x) \cdot b(x)$$

$$a(x) = c(x) + d(x)$$

$$b(x) = \cos(d(x))$$

$$c(x) = 4$$

$$d(x) = \sin(x)$$



2. Introduction of Derivative graph

$$f(x) = (4 + \sin(x)) \cdot \cos(\sin(x)) = a(x) \cdot b(x)$$

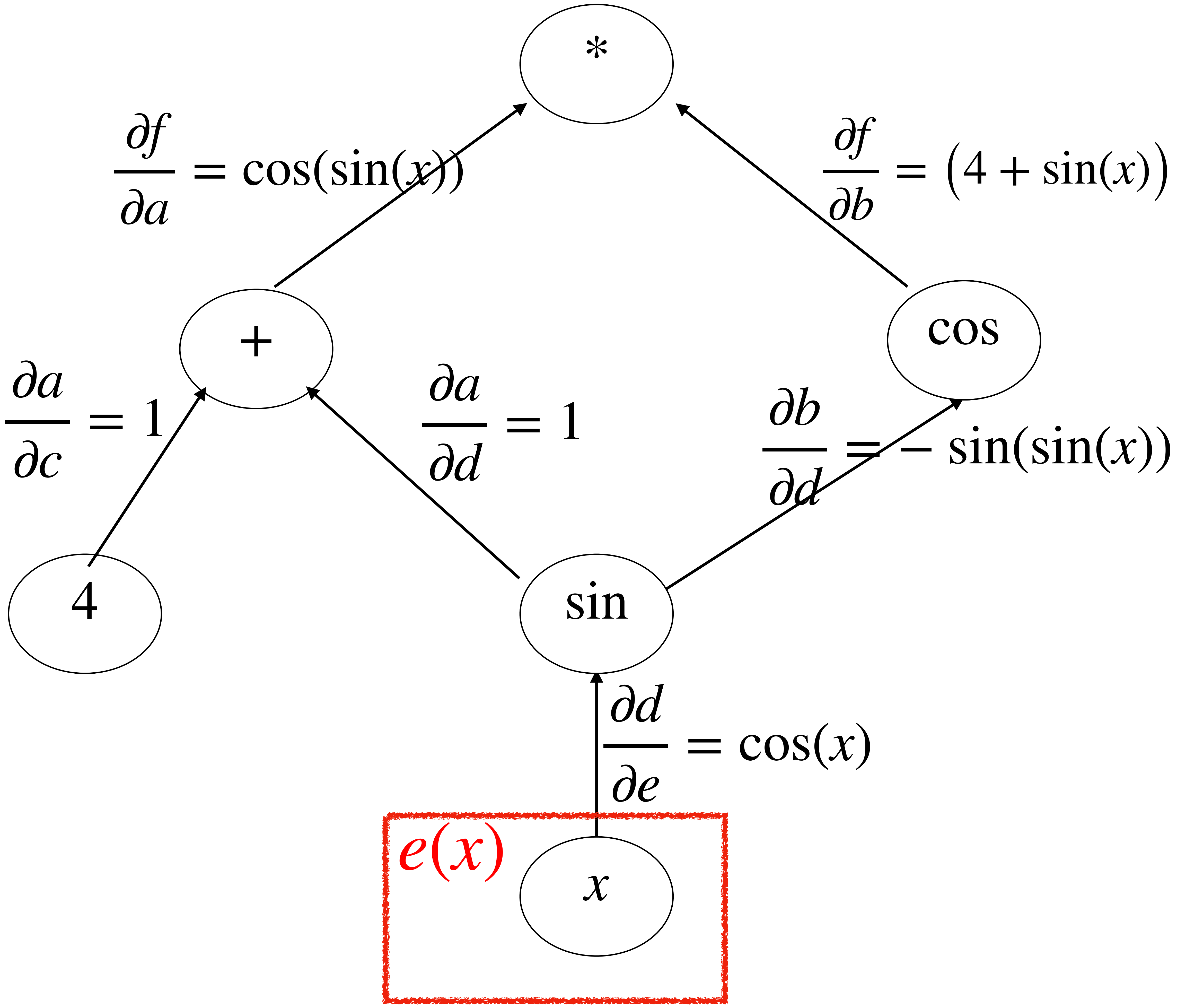
$$a(x) = c(x) + d(x)$$

$$b(x) = \cos(d(x))$$

$$c(x) = 4$$

$$d(x) = \sin(e(x))$$

$$e(x) = x$$



Evaluation of Derivative on Derivative Graph

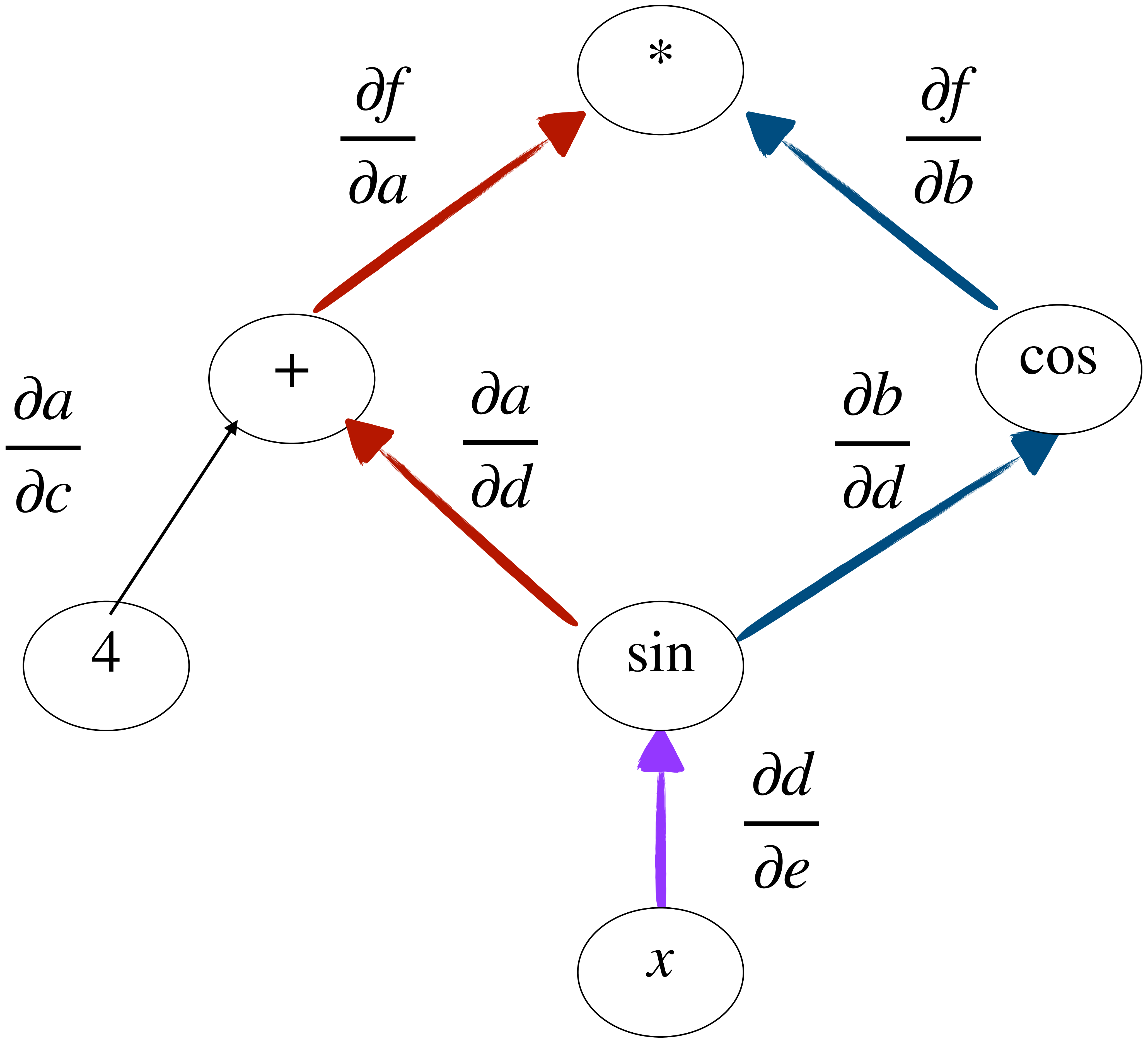
By using **chain rule**

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial f}{\partial b}\frac{\partial b}{\partial x}$$

$$= \frac{\partial f}{\partial a}\left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial a}{\partial d}\frac{\partial d}{\partial x}\right) + \frac{\partial f}{\partial b}\frac{\partial b}{\partial d}\frac{\partial d}{\partial x}$$

$$= \frac{\partial f}{\partial a}\left(\frac{\partial a}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial a}{\partial d}\frac{\partial d}{\partial e}\frac{\partial e}{\partial x}\right) + \frac{\partial f}{\partial b}\frac{\partial b}{\partial d}\frac{\partial d}{\partial e}\frac{\partial e}{\partial x}$$

$$= \underline{\frac{\partial f}{\partial a}\frac{\partial a}{\partial d}\frac{\partial d}{\partial e}} + \underline{\frac{\partial f}{\partial b}\frac{\partial b}{\partial d}\frac{\partial d}{\partial e}} \quad \left(\frac{\partial e}{\partial x} = 1, \frac{\partial c}{\partial x} = 0\right)$$



Sum of Product of paths from x to root

If you store values instead of expressions, it'll be **Automatic Differentiation.**

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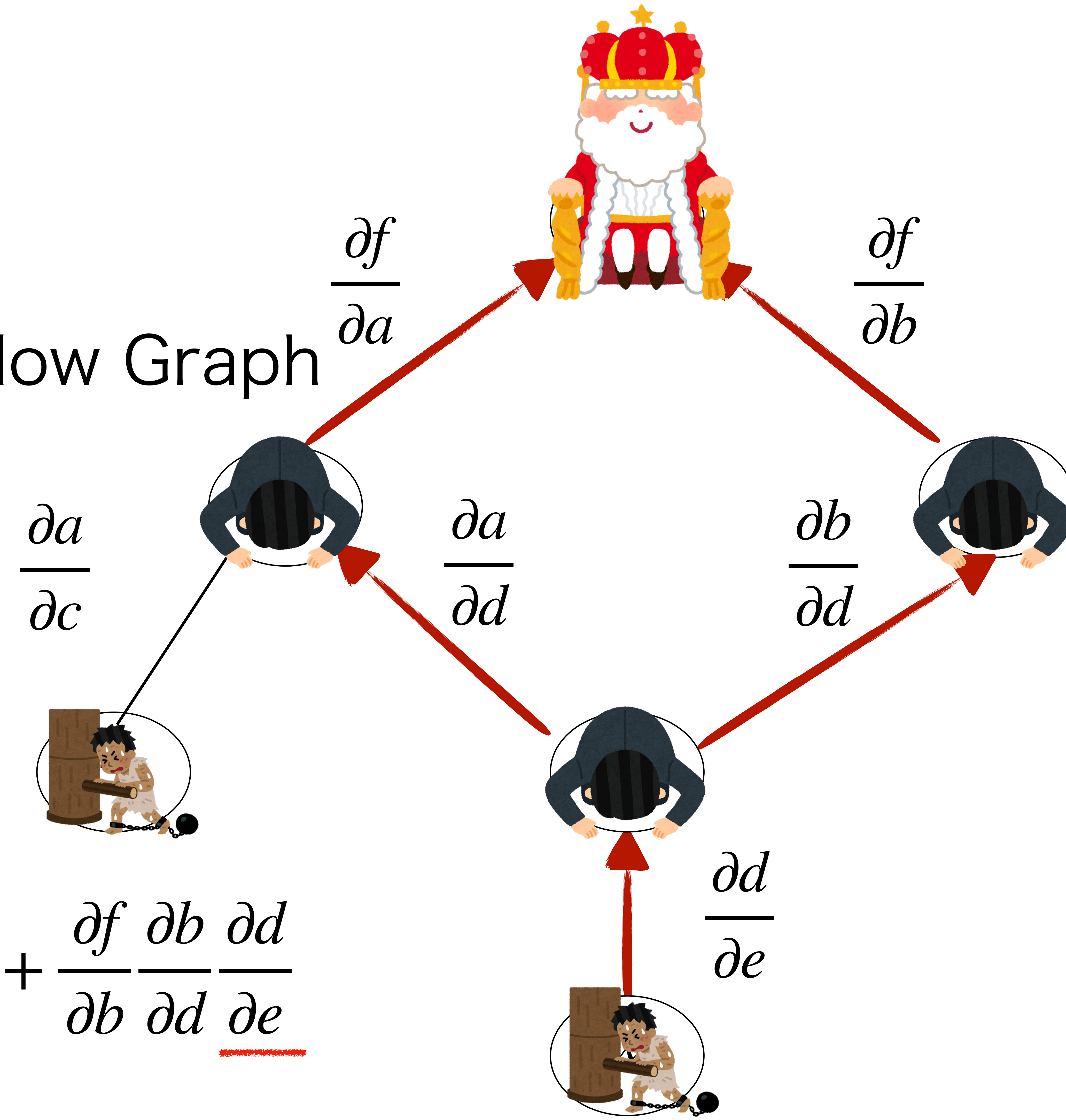
- Symbolic Differentiation and its improvement
- **Optimization of Derivative Evaluation**

Optimization of Derivative Evaluation[Guenter, SIGGRAPH2007]

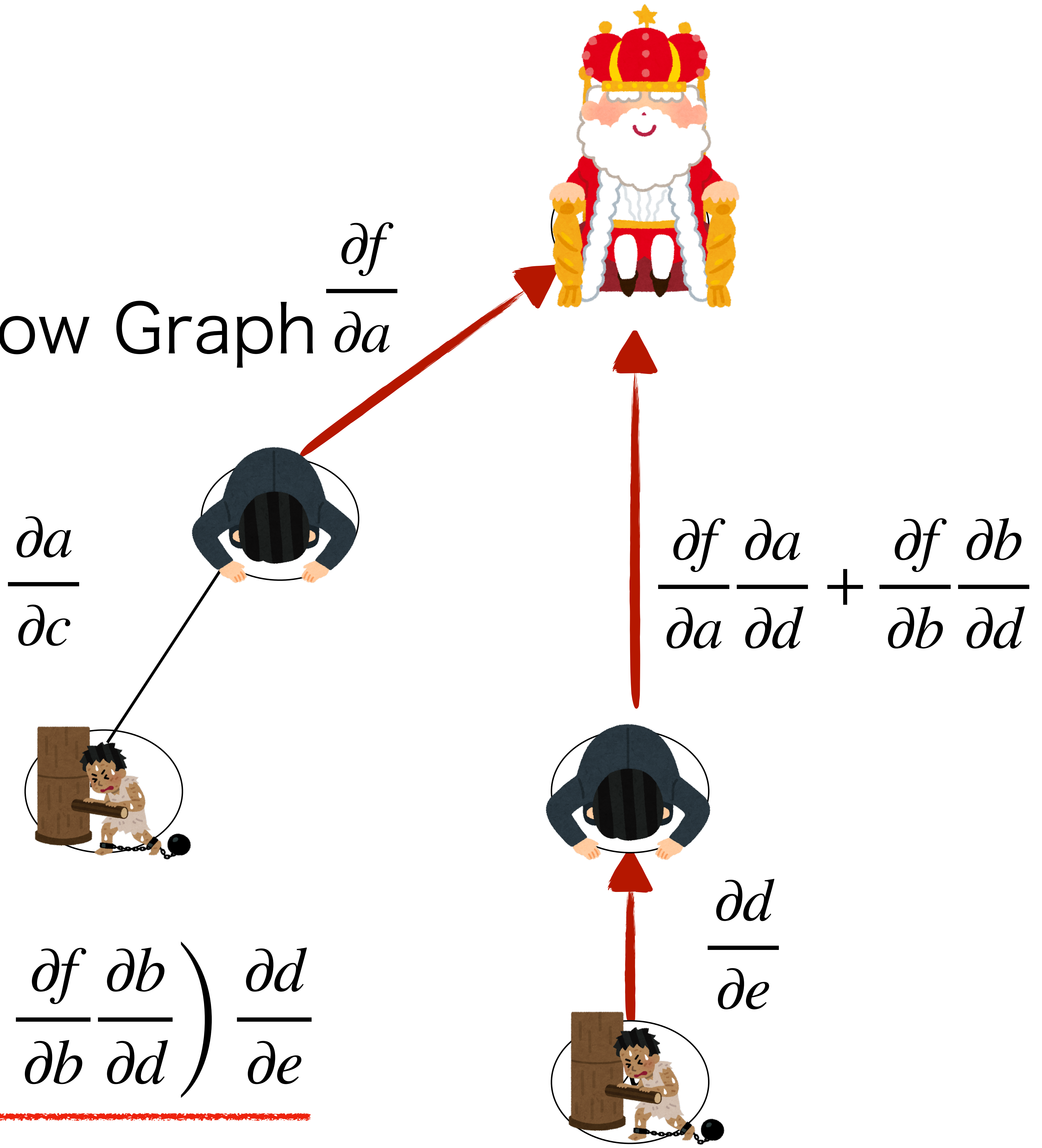
- Make Derivative Graphs small
  by using Dominance Relation of Control Flow Graph



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial d}\underline{\frac{\partial d}{\partial e}} + \frac{\partial f}{\partial b}\frac{\partial b}{\partial d}\underline{\frac{\partial d}{\partial e}}$$

Optimization of Derivative Evaluation[Guenter, SIGGRAPH2007]

- Make Derivative Graphs small
  by using Dominance Relation of Control Flow Graph
  - Performance improvement could be expected
    for repetitive evaluation.



$$\frac{\partial f}{\partial x} = \underline{\left(\frac{\partial f}{\partial a}\frac{\partial a}{\partial d} + \frac{\partial f}{\partial b}\frac{\partial b}{\partial d}\right)\frac{\partial d}{\partial e}}$$

Reduce 1 multiplication

Experimental Results

- Compare three Symbolic Differentiation methods https://github.com/khei4/sym_diff

```
evaluation times of single-variable derivative in 5 sec
naive expression tree walk: 2995 times
derivative graph: 3989 times
optimized derivative graph: 5925 times
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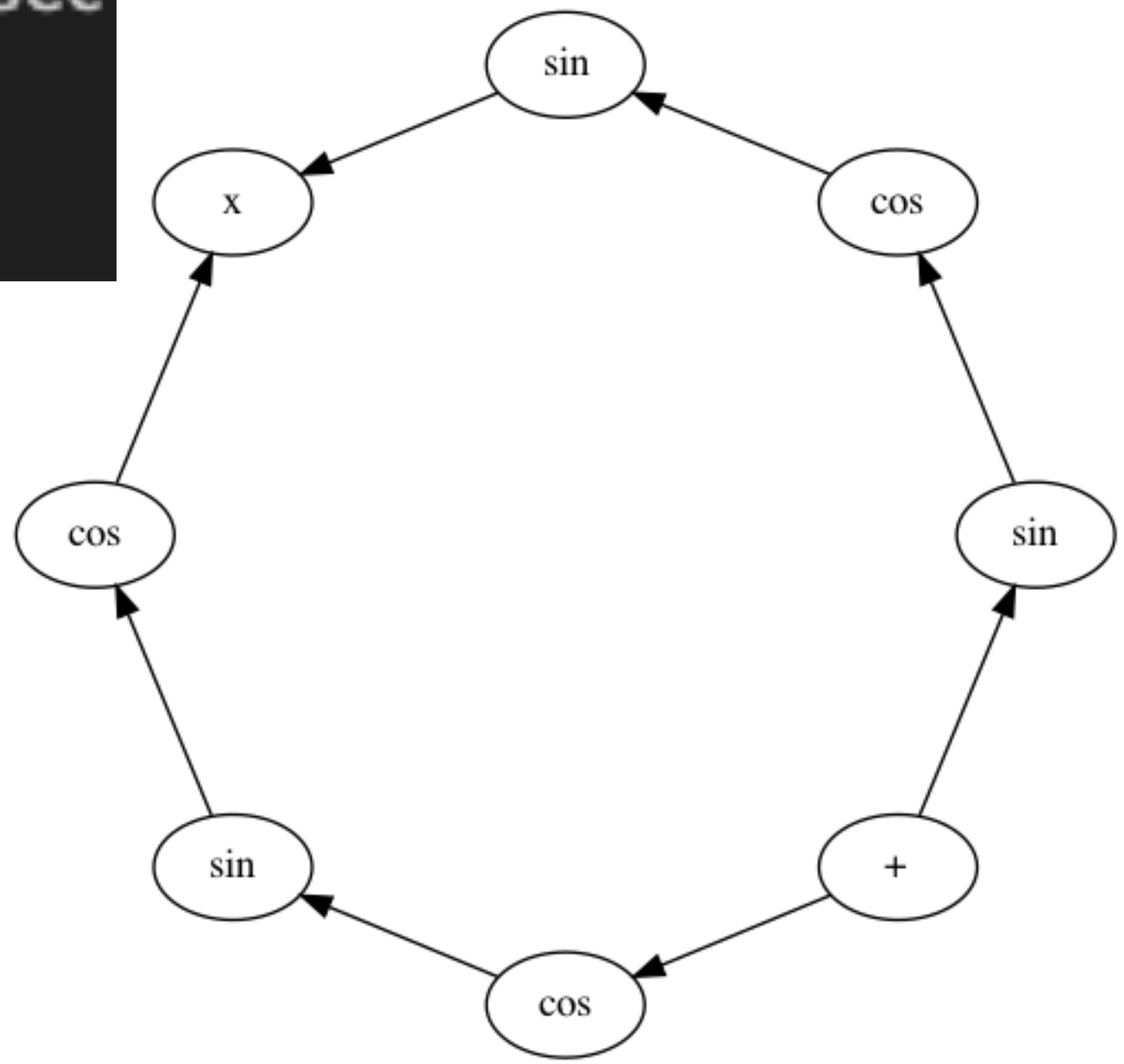

- Original paper shows this method exceeds Automatic Differentiation(CppAD) for repetitive evaluation.

Summary

- Symbolic Differentiation Improvement
  can naturally derive **Automatic Differentiation**.

- Derivative Evaluation can be optimized by using **Dominance Relation**